Proceedings of the 3rd International Seminar of Basic Sciences
©2017 FMIPA Pattimura University, Ambon, Indonesia
ISBN: 978-602-61644-0-7
pp. 149 – 157

THE TOTAL IRREGULARITY STRENGTH OF SOME COMPLETE BIPARTITE GRAPHS

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ABSTRACT

This paper deals with the total irregularity strength of complete bipartite graph $K_{m,n}$ where $2 \leq m \leq 4$ and $n > m$.

Keywords: *Complete bipartite graph; Total irregularity strength; Totally irregular total labeling*

1. INTRODUCTION

Let a graph G considered here be a finite, simple, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any total labeling $f: V(G) \cup E(G) \to \{1, 2, \dots, k\}$, the weight of a vertex v and the weight of an edge $e = xy$ are defined by $w(v) = f(v) + \sum_{uv \in E} f(uv)$ and $w(xy) = f(x) + f(y) + f(xy)$, respectively. If all the vertex weights under total labeling f are distinct, then f is called a vertex irregular total k-labeling, and if all the edge weights under total labeling f are distinct, then f is called an edge irregular total k-labeling. The minimum value of k for which G has a vertex (or an edge) irregular total labeling f is called the total vertex (or edge, resp.) irregularity strength of G and is denoted by $tvs(G)$ (or $tes(G)$, resp.) [1]. Baca, Jendrol, Miller, & Ryan in [1] gave the boundary for the $tvs(G)$ that for every (p, q)-graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, as follow.

$$\left\lceil \frac{p+\delta(G)}{\Delta(G)+1} \right\rceil \leq tvs(G) \leq p + \Delta(G) - 2\delta(G) + 1; \quad (1.1)$$

and for the total edge irregularity strength of graph G, as follow.

$$\left\lceil\frac{|E(G)|+2}{3}\right\rceil \leq tes(G) \leq |E(G)|. \quad (1.2)$$

Later, Jendrol *et al.* [2] determined the total edge-irregular strengths of a complete bipartite graph $K_{m,n}$, where $m, n \geq 2$, as follow.

$$tes(K_{m,n}) = \left\lceil\frac{mn+2}{3}\right\rceil. \quad (1.3)$$

For further results on tvs and tes, one can refer to [3].

In 2012, Marzuki, Salman, and Miller [4] introduced a new parameter by combining the vertex irregular total labeling and the edge irregular total labeling. A total k-labeling $f: V \cup E \to \{1, 2, \dots, k\}$ of G is called a *totally irregular total k-labeling* if for any pair of vertices x and y, their weights $w(x)$ and $w(y)$ are distinct and for any pair of edges x_1x_2 and y_1y_2, their weights $w(x_1x_2)$ and $w(y_1y_2)$ are distinct. The minimum value k for which a graph G has totally irregular total labeling, is called the *total irregularity strength* of G, denoted by $ts(G)$. They [4] have proved that for every graph G,

$$ts(G) \geq \max\{tes(G), tvs(G)\} \quad (1.4)$$

and determined the exact value of total irregularity strength of paths and cycles. For path P_n of n vertices,

$$ts(P_n) = \begin{cases} \left\lceil\frac{n+2}{3}\right\rceil, & \text{for } n \in \{2,5\}; \\ \left\lceil\frac{n+1}{2}\right\rceil, & \text{otherwise.} \end{cases} \quad (1.5)$$

In [5], Ramdani and Salman determined the ts of several cartesian product graphs. Later, Ramdani *et al.* [6] determined the ts for gear graphs, fungus graphs, $ts(Fg_n)$, for n even, $n \geq 6$; and for disjoint union of stars. Tilukay *et al.* in [7] determined the ts of fan, wheel, triangular book, and friendship graphs.

The result of the total irregularity strength of star graph $K_{1,n}$ is given by Indriati, et al. in [8]. They [8] obtained that for any positive integer $n \geq 3$,

$$ts(K_{1,n}) = \left\lceil\frac{n+1}{2}\right\rceil. \quad (1.6)$$

Recently, Tilukay *et al*. in [9] determined the ts of complete graph K_n and complete bipartite graph $(K_{n,n})$. They [9] obtained that for any positive integer $n \geq 2$,

$$ts(K_{n,n}) = \left\lceil \frac{n^2+2}{3} \right\rceil. \quad (1.7)$$

Completing the result above, in this paper, we give some more results by determining the total irregularity strength of complete bipartite graph $K_{m,n}$ where $2 \leq m \leq 4$ and $n > m$.

2. MAIN RESULT

Let $K_{m,n}$, where $m, n \geq 0$, be a complete bipartite graph with partite sets of cardinalities m and n. For simplifying the drawing of $K_{m,n}$ together with labels, let the labeling $f: V(K_{m,n}) \cup E(K_{m,n}) \to \{1, 2, \dots, k\}$ represented by an $(m+1) \times (n+1)$ matrix $M_f(K_{m,n}) = (\alpha_{ij})$, where $\alpha_{11} = 0$; first column $\alpha_{i1}, i \neq 1$ consists of labels of m vertices in second partite; first row $\alpha_{1i}, i \neq 1$ consists of labels of n vertices in first partite; and the rests consist labels of edges joining these vertices.

Theorem 1. Let $K_{m,n}$ be a complete bipartite graph with $2 \leq m \leq 4$ and $n > m$. Then

$$ts(K_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil.$$

Proof. Since $|V(K_{m,n})| = m + n$, $|E(K_{m,n})| = mn$, $\delta(G) = m$, $\Delta(G) = n$ with $m \leq n$ by (1.1), (1.2), (1.3) and (1.4), we have

$$ts(K_{m,n}) \geq \left\lceil \frac{mn+2}{3} \right\rceil. \quad (2.1)$$

For the reverse inequality, we construct an irregular total labeling $f: V \cup E \to \{1, 2, \dots, k\}$ which will be divided in two cases. Let $k = \left\lceil \frac{mn+2}{3} \right\rceil$.

Case 1. For $m = 2$;

Subcase 1.1. For $m = 2$ and $n \in \{3, 4, 5\}$;

The labeling are shown in Figure 1. It is easy to check that all edge-weights form arithmetic progression and all vertex-weights are distinct.

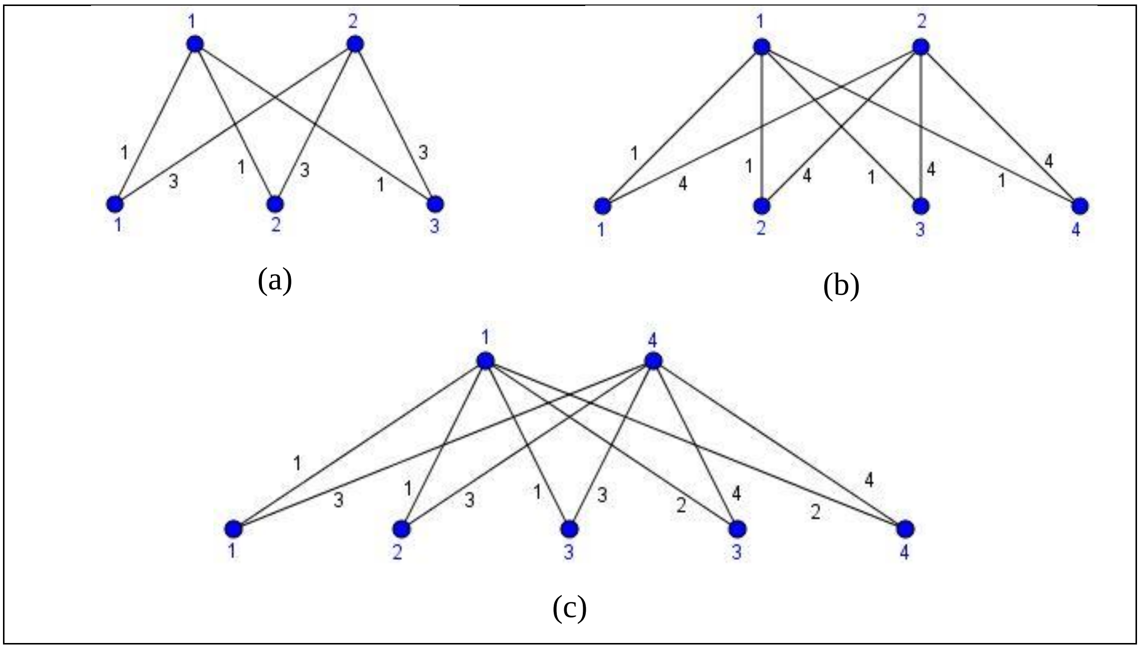

Figure 1. (a) Totally irregular total 3-labeling of $K_{2,3}$, **(b)** Totally irregular total 4-labeling of $K_{2,4}$, and **(c)** Totally irregular total 4-labeling of $K_{2,5}$

Thus, we have $M_f(K_{2,3}) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 \end{pmatrix}$; $M_f(K_{2,4}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 4 & 4 & 4 \end{pmatrix}$; and $M_f(K_{2,5}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 4 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}$.

Subcase 1.2. For $m = 2$ and $n \geq 6$;

Let $V(K_{2,n}) = \{u_i | 1 \leq i \leq n\} \cup \{x_1, x_2\}$ and $E(K_{2,n}) = \{u_i x_1,\ u_i x_2\ | 1 \leq i \leq n\}$.

Define

$$f(u_i) = \begin{cases} i, & \text{for } 1 \leq i \leq k; \\ i - n + k, & \text{for } k + 1 \leq i \leq n; \end{cases}$$

$$f(x_1) = 1;$$

$$f(x_2) = k;$$

$$f(u_i x_1) = \begin{cases} 1, & \text{for } 1 \leq i \leq k; \\ n - k + 1, & \text{for } k + 1 \leq i \leq n; \end{cases}$$

$$f(u_i x_2) = \begin{cases} n - k + 2, & \text{for } 1 \leq i \leq k; \\ 2(n - k + 1), & \text{for } k + 1 \leq i \leq n. \end{cases}$$

It is easy to check that the largest label is k.

Next, we verify the edge-weight and the vertex-weight set as follows.

For the edge-weight,

$w(u_i x_1) = i + 2$, for $1 \leq i \leq n$;

$w(u_i x_2) = n + i + 2$, for $1 \leq i \leq n$.

It can be checked that the weights of the edges under f are $3, 4, \cdots, mn + 2$.

For the vertex-weight,

$$w(u_i) = \begin{cases} n - k + i + 3, & \text{for } 1 \leq i \leq k; \\ 2n - 2k + i + 3, & \text{for } k + 1 \leq i \leq n; \end{cases}$$

$w(x_1) = (n - k)(n - k + 1) + k + 1$;

$w(x_2) = k(n - k + 3) + (2n - 2k)(n - k + 1)$.

It can be checked that there are no two vertices with same weight.

Case 2. $m = 3$;

Subcase 2.1. For $m = 2$ and $n \in \{5, 6\}$;

The labeling is given in matrices below.

$$M_f(K_{3,5}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \text{ and } M_f(K_{3,6}) = \begin{pmatrix} 0 & 1 & 2 & 4 & 5 & 6 & 7 \\ 1 & 1 & 2 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 2 & 3 & 4 & 5 \\ 7 & 7 & 7 & 6 & 6 & 6 & 6 \end{pmatrix}$$

It is easy to check that all edge-weights form a consecutive sequence $3, 4, \cdots, 3n + 2$ and all vertex-weights are distinct.

Subcase 2.2. For $m = 3$ and $n \notin \{5, 6\}$;

Let $V(K_{3,n}) = \{u_i, v_j | 1 \leq i \leq b \text{ and } 1 \leq j \leq a\} \cup \{x_i, y_1 | 1 \leq i \leq 2\}$ and

$E(K_{3,n}) = \{u_i x_j | 1 \leq i \leq b \text{ and } 1 \leq j \leq 2\} \cup \{v_i x_j | 1 \leq i \leq a \text{ and } 1 \leq j \leq 2\} \cup \{u_i y_1 | 1 \leq i \leq b\} \cup \{v_i y_1 | 1 \leq i \leq a\}$

Define,

$f(u_i) = i$ for $1 \leq i \leq b$

$f(v_j) = k - (a - j)$ for $1 \leq j \leq a$

$f(x_i) = i$ for $1 \leq i \leq 2$

$f(y_1) = k$

$f(u_i x_j) = i$ for $1 \leq i \leq b$ and $1 \leq j \leq 2$

$f(v_i x_j) = 2b + a - k + i$ for $1 \leq i \leq a$ and $1 \leq j \leq 2$

$f(u_i y_1) = k$ for $1 \leq i \leq b$

$f(v_j y_1) = k - 1$ for $1 \leq j \leq a$

It is easy to check that the largest label is k.

For the edge-weight,

$w(u_i x_j) = 2i + j$ for $1 \leq i \leq b$ and $1 \leq j \leq 2$

$w(v_i x_j) = 2(b + i) + j$ for $1 \leq i \leq a$ and $1 \leq j \leq 2$

$w(u_i y_1) = 2k + i$ for $1 \leq i \leq b$

$w(v_j y_1) = 3k - (a - j) - 1$ for $1 \leq j \leq a$

It can be checked that the weights of the edges under f are $3, 4, \cdots, mn + 2$.

For the vertex-weights,

$w(u_i) = 3i + k$ for $1 \leq i \leq b$

$w(v_j) = 4b + a + 3j - 1$ for $1 \leq j \leq a$

$w(x_i) = i + \frac{(1+b)b}{2} + \frac{(4b+3a-2k+1)a}{2}$ for $1 \leq i \leq 2$

$w(y_1) = k(n + 1) - a$

It can be checked that there are no two vertices with same weight.

Case 3. $m = 4$;

Subcase 3.1. For $m = 4$ and $n = 14$;

The labeling is given in matrix below.

$$M_f(K_{4,14}) = \begin{pmatrix} 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 10 & 12 & 14 & 16 & 18 & 20 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 8 & 8 & 8 & 8 & 8 \\ 19 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 18 & 18 & 18 & 18 & 18 & 18 \\ 20 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 18 & 18 & 18 & 18 & 18 & 18 \end{pmatrix}$$

Subcase 3.1. For $m = 4$ and $n \neq 14$;

Let $a = \left\lceil \frac{k}{2} \right\rceil$ and $b = n - \left\lceil \frac{k}{2} \right\rceil$ for $5 \leq n \leq 8$, $13 \leq n \leq 18$ or $n \geq 25$; let $a = \left\lfloor \frac{n}{2} \right\rfloor$ and $b = \left\lceil \frac{n}{2} \right\rceil$ for $9 \leq n \leq 12$ and $19 \leq n \leq 24$.

Let

$V(K_{4,n}) = \{u_i, v_j | 1 \leq i \leq a \text{ and } 1 \leq j \leq b\} \cup \{x_i, y_j | 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 2\}$ and

$$E(K_{4,n}) = \{u_i x_j | 1 \leq i \leq a, 1 \leq j \leq 2\} \cup \{v_i x_j | 1 \leq i \leq b, 1 \leq j \leq 2\} \cup \{u_i y_j | 1 \leq i \leq a, 1 \leq j \leq 2\} \cup \{v_i y_j | 1 \leq i \leq b, 1 \leq j \leq 2\}.$$

Define

$f(u_i) = 2i - 1,$ for $1 \leq i \leq a$;

$f(v_i) = k - 2b,$ for $1 \leq i \leq b$;

$f(x_i) = i,$ for $1 \leq i \leq 2$;

$f(y_i) = k + i - 2,$ for $1 \leq i \leq 2$;

$f(u_i x_j) = 1,$ for $1 \leq i \leq a$ and $1 \leq j \leq 2$;

$f(v_i x_j) = 2n - k,$ for $1 \leq i \leq b$ and $1 \leq j \leq 2$;

$f(u_i y_j) = 2n - k + 3,$ for $1 \leq i \leq a$ and $1 \leq j \leq 2$;

$f(v_i y_j) = 4n - 2k + 2,$ for $1 \leq i \leq b$ and $1 \leq j \leq 2$.

It is easy to check that the largest label is k.

For the edge-weight,

$w(u_i x_j) = 2i + j,$ for $1 \leq i \leq a$ and $1 \leq j \leq 2$;

$w(v_i x_j) = 2a + 2i + j,$ for $1 \leq i \leq b$ and $1 \leq j \leq 2$;

$w(u_i y_j) = 2n + 2i + j,$ for $1 \leq i \leq a$ and $1 \leq j \leq 2$;

$w(v_i y_j) = 4n - 2b + 2i + j,$ for $1 \leq i \leq b$ and $1 \leq j \leq 2$.

It can be checked that the weights of the edges under f are $3, 4, \cdots, 4n + 2$.

For the vertex-weights,

$w(u_i) = 4n - 2k + 2i + 7,$ for $1 \leq i \leq a$;

$w(v_i) = 12n - 5k - 2b + 2i + 4,$ for $1 \leq i \leq b$;

$w(x_i) = a + b(2n - k) + i,$ for $1 \leq i \leq 2$;

$w(y_j) = a(2n - k + 3) + b(4n - 2k + 2) + k + j - 2,$ for $1 \leq j \leq 2$.

It can be checked that $w(u_i)$ (and $w(v_j)$) under f form an arithmetic progression with difference 2 while $w(x_i)$ (and $w(y_j)$) under total labeling f form an arithmetic progression with difference 1, and there are no two vertices with same weight.

Based on the results of four cases, we can conclude that f is the totally irregular total $\left\lceil \frac{mn+2}{3} \right\rceil$-labeling. Thus, for $2 \leq m \leq 4$ and $n > m$, we obtained:

$$ts(K_{m,n}) \leq \left\lceil \frac{mn+2}{3} \right\rceil. \tag{2.2}$$

By, (2.1) and (2.2), we have $ts(K_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil$, for $2 \leq m \leq 4$ and $n > m$. ■

3. CONCLUSION

By Theorem 1, we can conclude that for $2 \leq m \leq 4$ and $n > m$,

$$ts(K_{m,n}) = \left\lceil \frac{mn+2}{3} \right\rceil.$$

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