APPLICATION PERSPECTIVE OF OPTICAL FIBER RAYS IN EIKONAL EQUATION

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Abstract

Characterization of ray trajectories in nonhomogeneous media is based on a traditional Eikonal equation valid in the region of geometrical optics, whereby the vector calculus and geometry are require fairly extensive. In this paper, we introduce a simple correspondence rule, and a few intuitive guidelines for relating rays to waves, an equivalent set of trajectory equations can be obtained. Our approach results in the Eikonal equations for the space evolution of optical fiber rays which are expressed in terms of the mode analysis invariants, azimuthal mode number, and optical waveguide propagation phase constant. It can expand application areas such as sensors and lightwave telecommunications.

Keywords – *Fiber Optics, Optical Geometry, Ray Trajectories*.

I. INTRODUCTION

An extensive body of literature exists for both wave and ray characterization of optical fibers. It is generally accepted that exact analityc waveguide solutions exist for a select number of specific index profiles. The exact waveguide solutions are frequently derived for step index fibers in intermediate level optical fiber texts [1], [2]. In the general case of graded index fibers the problems is not analytically tractable without approximation [1]. The most common technique for the analysis of graded index fibers is based on the Wentzel-Kramer-Brillouin (WKB) approximation [3]. It is possible to show that the first-order WKB approximation is equivalent to the results produced from a simple ray model [4].

Despite the mathematic importance of having an exact waveguide solution in an optical fiber, the propagation characteristics can be difficult to visualize. It is not surprising that an alternative approach to treating optical fibers, based on the Eikonal ray approach, which in turn is based on Fermat's extremum principle [5], has proven to be extremely useful in explaining effects which take place in optical fibers [2]. For example, application of such techniques to the prediction of bending losses in optical fibers has appeared in the literature [6]. In addition, achromatic modal dispersion effects have been predicted with ray models [7].

Recently, an approach based on a simple correspondence principle between waves and rays has been proposed as a substitute for the first-order WKB approach of counting modes [8]. This correspondence principle has been used to qualitatively demonstrate that there should be a direct association between the mode numbers in an optical fiber and the ray trajectories. The main focus of this brief report is to provide a simple approach for deriving the three dimensional (3D) dynamic trajectory equations for optical fibers using this correspondence principle. We have found that the application of these concepts in the classroom has proven useful in explaining principles of wave propagation within optical fibers. In particular, the physical interpretation of optical fiber modes as rays, which is emphazed here, complements the more rigorous waveguide analysis because it facilitates the ability to visualize modes. With that visualization comes a better physical understanding of many of the mode-dependent effects such as bending losses and modal dispersion.

II. THEORETICAL BACKGROUND

Starting from the Hemholtz equation which is the phasor form of the wave equation in linear, isotropic source-free homogeneous media, we have

$$\left[\nabla^2 + k^2(\vec{r})\right]\vec{E} = 0 \qquad (1)$$

where $k(\vec{r})$ is the wavenumber and $\vec{E}$ is the electric field. It is noted that this equation is only approximately valid in a graded index fiber. Consistent with the form given in (1) se assumed that

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$$k^2(r) = n^2(r)k^2 = k_r^2 + k_\phi^2 + k_z^2 \qquad (2)$$

where k is the vacuum wavenumber, $n(r)$ is the index of refraction, and r, ϕ and z are cylindrical coordinates needed to describe points in the fiber. Following a standar procedure [1] in waveguide analysis for circularly symmetric fibers it is assumed

$$E_z\,(r,\phi,z) = F_v(r)^{jv\phi} e^{-j\beta z} \qquad (3)$$

where $F_v(r)$ is the radial solution to source-free homogeneous Hemholtz equation, β is the waveguide phase constans of the waveguide, and v is the azimuthal mode number, which is forced to be an integer due to periodic boundary conditions. $F(r)$ is the radial solution to (1). In a step index fiber, the exact waveguide solution for $F(r)$ can be expresed on terms of Bessel function [1], [3]. To be precise, the source-free homogenous Hemholtz equationcannot exactly predict the waveguide solution in the graded index optical fiber. However, within the regime of geometrical optics the discrepancy is guaranteed to be small [3].

It is generally accepted that an arbitary electromagnetic field has an angular plane wave representation [9]. The view-point being developed her is based on the interpretation that only a plane wave can be described by a single ray and the direction of propagation of an optical ray is the propagation vector. A simple correspondence rule for plane waves [8]

$$\vec{\nabla} \rightarrow -j\vec{k} \qquad (4)$$

Describes the action of the Del operator in terms of an equivalent algebraic substitution involving the wavevector. We proceed to apply (4) to the ϕ and z dependencies of (3). After application of tile grad operator in cylindrical coordinates given by

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\phi \frac{1}{r}\frac{\partial}{\partial_\phi} + \hat{e}_z \frac{\partial}{\partial_z} \qquad (5)$$

it follows from (5) that

$$j(\vec{\nabla} E_z)_\phi = j\frac{v}{r}E_z \qquad (6)$$

which in combination with (4) leads to

$$k_\phi = -\frac{v}{r} \qquad (7)$$

a correspondence rule for the ϕ component of the propagation vector. Similarly, from (5)

$$j(\vec{\nabla} E_z)_z = -j\beta E_z \qquad (8)$$

and again from (4)

$$k_z = \beta \qquad (9)$$

which is the z-component correspondence rule. It is noted from (3) that the correspondence rule (4) applies only to the ϕ and the z components.

For radially symmetric index profiles, Snell' law requires that β and v are constants of the motion. It is known that the partial fields associated with incident, transmitted, and reflected waves at a boundary must exhibit temporal and spatial phase synchronization. This requires that the tangential component of the propagation vector is continuous across boundaries [10]. In particural, under the assumption of a radially symmetric profile the surfaces of constant index of refraction are cylinders. Applyng the above rule this requires for the ϕ component of the propagation vector.

$$k_{\phi_1} = k_{\phi_2} \qquad (10)$$

and therefore according to (7) $v_1 = v_2 = v$. Similiary, for the z component of the propagation vector

$$k_{z_1} = k_{z_2} \qquad (11)$$

and therefore according to (9), $\beta_1 = \beta_2 = \beta$. To put this in perspective let $\beta = kn(r)\cos\theta$, which reflects the interpretation that the propagation constant β is the z component of the propagation vector seen on Fig. 1. Then (11) is directly equivalent to more commonly seen optic form for Snell's law $n_1 \cos\phi_1 = n_2 \cos\phi_2$.

Using the relation between the propagation vector and its component, it follow fi'om (7) and (8) that the radial component of the propagation can be expressed as

$$k_r = \pm\sqrt{k^2(r) - \beta^2 - \left(\frac{v}{r}\right)^2} \qquad (12)$$

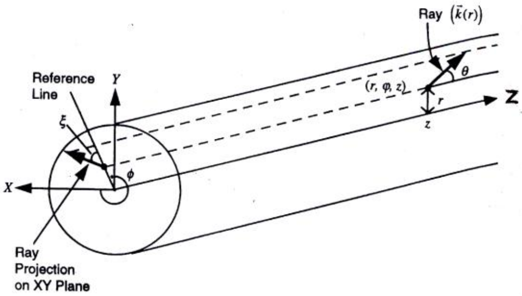

Fig. l. Geometry defining location and orientation of the wavevector in an optical fiber.

The space dependence of the wave number is defined in terms of the index of refraction

$$k(r) = k_n(r) \qquad (13)$$

where k is the vacuum wavenumber. Both +/- roots of (8) are required to characterize a full cycle of the ray trajectory.

III. RAY TRAJECTORY EQUATIONS IN CYLINDRICAL COORDINATES

In this section, the equations which describe the ray trajectories will be derived. The basic principle involved is that in isotropic media the wavevector describes the normal to the wavefront. Making the natural association between wavefront normal and rays, the resulting principles is the ray trajectory tangents are parallel to the wavevector.

With reference to Fig. 1, the orientantion of the wavector wavevector $k(r)$ is defined in terms of θ and ϕ. It follows that

$$k\phi = k(r)\sin\phi\sin\xi \qquad (14)$$

$$k_z = k(r)\cos\phi \qquad (15)$$

and

$$k_r = k(r)\sin\phi\cos\xi \qquad (16)$$

The incremental line variation in the ray trajectory can be defined as

$$\vec{d}\ell = dz\hat{e}_z + dr\hat{e}_r + rd\phi\hat{e}_\phi \qquad (17)$$

The wavevector, defined by the cylindrical components (14)-(16), is perpendicular to the wavefront of constant phase the therefore tangent to the trajectories. The incremental component $d\phi, dr,$, and dz are tangential to the wavevector defined by component (7), (9), and (12), and therefore by comparison with (14)-(16)

$$dz = d\ell\cos\phi \qquad (18)$$

$$dr = d\ell\sin\phi\cos\xi \qquad (19)$$

and

$$rd\phi = d\ell\sin\phi\sin\xi \qquad (20)$$

defined the ray tangent.

Through combinations of (9), (12), (15), (16), (18), And (19) we have that

$$\frac{dz}{dr} = \frac{k_z}{k_r} = \frac{\beta}{k_r} = \pm\frac{\beta}{\sqrt{k^2 n^2(r) - \beta^2 - \left(\frac{v}{r}\right)^2}} \qquad (21)$$

Working similarly with the ϕ component of the ray, combination of (14), (16), (19), and (20) gives

$$r\frac{d\phi}{dr} = \frac{k_\phi}{k_r} = \pm\frac{-\frac{v}{r}}{\sqrt{k^2 n^2(r) - \beta^2 - \left(\frac{v}{r}\right)^2}} \qquad (22)$$

Which is the evolution for the ϕ component of the ray. Equations (21) and (22), which describe the ray trajectories, are shown to be in agreement with the Eikonal solution [2] in the next section. Unlike the standard Eikonal evolution equations, (21) and (22) are expressed in terms of the azimuthal mode number and the waveguide phase constant. The developments in this section are limited because little insighs is provided as to what, if any, are the constraints on allowed values of ν and β for and optical fiber.

IV. A COMPARISON TO EIKONAL ANALYSIS AND APPLICATION PERSPECTIVE

The proposed evolution equations that describe the ray trajectories in the fiber optic according to the EikonaI approach [2] are

$$\frac{dr}{dz} = \left(\frac{n^2(r)}{E} - \frac{\ell^2}{r^2} - 1 \right)^{1/2} \tag{23}$$

and

$$\frac{d\phi}{dz} = \frac{\ell}{r^2} \tag{24}$$

where E dan ℓ are constant of the motion defined as

$$E = n(r)\frac{dz}{ds} \tag{25}$$

and

$$\ell = r^2 \frac{d\phi}{dz} \tag{26}$$

with s being the distance of the point measured along the ray trajectory. All other parameters appearing in (23)-(26) are defined in the previous discussion and in Fig 1. In what follows we demonstrate the relation between the Eikonal constants of motion and the invariants β and ν.

From the definition of $d\ell$ defined by (17) it follows that

$$ds = d\ell \tag{27}$$

By substituting (27) and (18) into (25), we have

$$E = n(r)\cos\phi \tag{28}$$

which, through combinations of (11) and (15), becomes

$$E = \frac{\beta}{k} \tag{29}$$

which relates the phase constant β to Eikonal constant *E*. Also substitution of (19) and (20) into (26) gives

$$\ell = r\frac{\sin\phi \sin\ell}{\cos\phi} \tag{30}$$

which, when combined with (7), (9), (14), and (15), leads to

$$\ell = -\frac{\nu}{\beta} \tag{31}$$

Equation (31) relates the azimuthal mode number ν and the phase constant β to the Eikonal constant ℓ.

First substitution of (29) and (30) into (23) leads to (21). Second substitution of (31) into (24) and (31) into the form

$$\frac{dr/dz}{d\phi/dz} \tag{32}$$

leads to (22). Therefore, there is an exact agreement (32) between the analysis presented in the previous section and the Eikonal analysis. More importantly, the proposed analysis connects the standard Eikonal invariants to the optical fiber mode-analysis invariants, without requiring extensive vector calculus. Because of this link, the numeric implementation of the model can provide valuable physical insight for the fiber optic application areas such as sensors or lightwave telecommunications [11].

V. CONCLUSION

The main point of this paper on optical fiber ray trajectories is to demonstrate the existence of a pedagogically attractive alternative to more formal methods such as WKB or Eikonal analysis. An important feature of the method presented is that it links the cylindrical waveguide invariants to the ray trajectories. This link can expan sensors and lightwave telecommunications application at near the future.

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